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An L^p inequality for 'self-reciprocal' polynomials

M.A. Qazi

Department of Mathematics, Tuskegee University, Tuskegee, AL 36088, USA

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Abstract

A polynomial f of degree at most n is said to be 'self-reciprocal' if $f(z) \equiv z^n f(1/z)$. In this paper we discuss the growth of the integral means of such polynomials on $|z| = R$, as $R \to \infty$.

Keywords: Self-reciprocal polynomials; L^p inequalities; Integral means

1. Introduction and statement of results

Let $\mathcal{P}_n$ be the class of all polynomials of degree at most n. Polynomials f in $\mathcal{P}_n$, which satisfy the condition $f(z) \equiv z^n f(1/z)$ have been studied since about thirty years (see [1–3], [4, §7.5], [5–8,12], [13, pp. 229–230], [14, pp. 431–432]). Frappier, Rahman and Ruscheweyh [4, p. 96] call such a polynomial 'self-reciprocal.' Let $\wp_n$ denote the sub-class of $\mathcal{P}_n$ consisting of 'self-reciprocal' polynomials. Thus, $f \in \wp_n$ if and only if $f \in \mathcal{P}_n$ and $f(z) \equiv z^n f(1/z)$. Ostensibly, the identically zero polynomial satisfies these conditions but we exclude it from the class $\wp_n$, once for all.

For any entire function F, let

$$\mathcal{M}_p(F;r) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(re^{i\theta}\right)\right|^p \,d\theta\right)^{1/p} \quad (0 < p < \infty;\ r > 0). \tag{1}$$

E-mail address: qazima@aol.com.

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It is well known (see, for example, [10, p. 143]) that for any given $r > 0$,

$$\mathcal{M}_p(F;r) \to \max_{|z|=r} |F(z)| \quad \text{as } p \to \infty.$$

Hence,

$$M(F;r) := \max_{|z|=r} |F(z)| \quad (r > 0), \tag{2}$$

may be seen as $\mathcal{M}_\infty(F;r)$. It is also known (see [10, p. 139]) that for any given $r > 0$,

$$\mathcal{M}_p(F;r) \to \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|F\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta\right) \quad \text{as } p \to 0.$$

So, we set

$$\mathcal{M}_0(F;r) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|F\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta\right). \tag{3}$$

When F is a polynomial, the associated quantity $\mathcal{M}_0(F;1)$ is called its *logarithmic Mahler measure*. Let $F(z) := a_m \prod_{\mu=1}^{m}(z - z_\mu)$, and suppose that $F(0)$ is not zero. Then by Jensen's theorem (see [15, p. 124])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|F\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta = \log \frac{|F(0)|}{\prod_{|z_\mu|<1} |z_\mu|}.$$

Hence

$$\mathcal{M}_0(F;1) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|F\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta\right) = |a_m| \prod_{\mu=1}^{m} \max\{|z_\mu|, 1\}. \tag{4}$$

Let $F(z) := z^m \Phi(z)$, where $\Phi(0) \neq 0$. Then $\mathcal{M}_0(F;1) = \mathcal{M}_0(\Phi;1)$, and (4) applies to Φ. Hence, in (4) the restriction '$F(0) \neq 0$' may be dropped, making it a very useful formula.

There exists (see [4, p. 97]) a polynomial $f \in \wp_n$ such that

$$M(f';1) \geqslant (n-1)M(f;1). \tag{5}$$

This is probably the most significant fact known about the class $\wp_n$. If $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is such that $a_0 = a_n$ then (see [5, Theorem 2])

$$M(f';1) \leqslant \left(n - \frac{1}{2} + \frac{1}{2(n+1)}\right) M(f;1). \tag{6}$$

In particular, (6) holds for any $f \in \wp_n$. It would be interesting to find the exact value of

$$A_n := \sup\{M(f';1)\colon\ f \in \wp_n,\ M(f;1) = 1\}.$$

For any $r > 0$, let

$$B_n(r) := \sup\{M(f;r)\colon\ f \in \wp_n,\ M(f;1) = 1\}. \tag{7}$$

Clearly,

$$B_n(r) = r^n B_n\left(\frac{1}{r}\right) \quad (r > 0).$$

It may be noted that f belongs to $\wp_1$ if and only if $f(z)$ is a constant multiple of $z+1$, and so $B_1(R) = (R+1)/2$ for any $R > 1$. Although not trivial, it is known (see [13, pp. 229–230]) that

$$B_2(R) = \frac{R^2+1}{2} \quad (R > 1). \tag{8}$$

For any integer $n \geqslant 3$, there exists (see [14, pp. 431–432]) a polynomial $f \in \wp_n$ such that

$$M(f; R) \geqslant \frac{R^n + R^{n-2}}{2} M(f; 1) \quad (R > 1),$$

and thus

$$B_n(R) \geqslant \frac{R^n + R^{n-2}}{2} \quad (R > 1;\ n = 3, 4, \ldots). \tag{9}$$

The reader will find upper bounds for $B_n(R)$ in [3], but the exact value of $B_n(R)$ remains unknown for $n = 3, 4, \ldots.$

Frappier and Rahman [3] also looked for the infimum of $\frac{M(f;R)}{M(f;1)}$ over all polynomials f belonging to $\wp_n$. They proved [3, Theorem 9] the following result.

Theorem A. *Let*

$$\psi(t) := \frac{1 + 2\sum_{\nu=1}^{\infty} t^{-2\nu^2}}{2\sum_{\nu=0} t^{-(2\nu+1)^2/2}} \quad (t > 1).$$

Then, for any $f \in \wp_n$ and any $R > 1$, we have

$$\frac{\mathcal{M}_\infty(f; R)}{\mathcal{M}_\infty(f; 1)} = \frac{M(f; R)}{M(f; 1)} \geqslant \begin{cases} R^{n/2} & \text{if } n \text{ is even,} \\ R^{n/2}\psi(R) & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

Let $p \in [0, \infty)$ and $R \in (1, \infty)$ be given. Here, in analogy with Theorem A, we look for the infimum of $\mathcal{M}_p(f; R)/\mathcal{M}_p(f; 1)$ as f is allowed to vary in $\wp_n$. Our first result deals with the case $p = 0$.

Theorem 1. *Let $\mathcal{M}_0(F; r)$ be as in* (3). *Then, for any $f \in \wp_n$ and any $R > 1$, we have*

$$\mathcal{M}_0(f; R) \geqslant \mathcal{M}_0(f; 1) \begin{cases} R^{n/2} & \text{if } n \text{ is even,} \\ R^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

The estimate is sharp.

In the case where $p \in (0, \infty)$, we prove the following result.

Theorem 2. *For any $p \in (0, \infty)$, let $\mathcal{M}_p(F; r)$ be as in* (3). *Then, for any $f \in \wp_n$, we have*

$$\mathcal{M}_p(f; R) \geqslant R^{n/2} \mathcal{M}_p(f; 1) \quad (R > 1). \tag{12}$$

The estimate is sharp for any $p \in (0, \infty)$ and any $R > 1$ in the case where n is even.

Remark. Let n be odd, and take any polynomial f belonging to $\wp_n$. Then $g(z) := f(z)/(z+1)$ is a polynomial of degree $n-1$ belonging to $\wp_{n-1}$. Applying (12) to g we see that for $0 < p < \infty$ and any $R > 1$, we have

$$\begin{aligned}\mathcal{M}_p(f;R) &\geqslant (R-1)\mathcal{M}_p(g;R) \geqslant (R-1)R^{(n-1)/2}\mathcal{M}_p(g;1)\\ &\geqslant \frac{R^{(n+1)/2}-R^{(n-1)/2}}{2}\mathcal{M}_p(f;1).\end{aligned}$$

Thus, in the case where n is odd, inequality (12) can be replaced by

$$\mathcal{M}_p(f;R) \geqslant \max\left\{R^{n/2}, \frac{R^{(n+1)/2}-R^{(n-1)/2}}{2}\right\}\mathcal{M}_p(f;1), \tag{12$*$}$$

without any hassle. However, (12∗) is not sharp for any $R > 1$. The following result contains the precise version of (12∗) for $p = 2$.

Theorem 3. *Let n be odd. Then for any $f \in \wp_n$, we have*

$$\mathcal{M}_2(f;R) \geqslant \left(\frac{R^{n+1}+R^{n-1}}{2}\right)^{1/2}\mathcal{M}_2(f;1) \quad (R > 1). \tag{13}$$

The estimate is sharp for any $R > 1$.

2. Proofs

Proof of Theorem 1. For $z = -1$ the condition $f(z) \equiv z^n f(1/z)$ reduces to $f(-1) = (-1)^n f(-1)$ and so to $f(-1) = -f(-1)$ if n is odd. Hence, a polynomial $f \in \wp_n$ must necessarily vanish at -1 in the case where n is odd. This fact is to be kept in mind since it makes all the difference when n is odd.

Since $f(z) \equiv z^n f(1/z)$, the multiplicity of any zero the polynomial f might have at the origin cannot exceed $n/2$ if n is even and $(n-1)/2$ if n is odd. Hence f must have the form

$$f(z) := \sum_{\nu=\ell}^{n-\ell} a_\nu z^\nu = z^\ell \sum_{\nu=\ell}^{n-\ell} a_\nu z^{\nu-\ell},$$

where $a_\ell = a_{n-\ell} \neq 0$ and $0 \leqslant \ell \leqslant n/2$ if n is even and $0 \leqslant \ell \leqslant (n-1)/2$ if n is odd. Let $\varphi(z) := z^{-\ell} f(z)$.

First let us suppose that $\varphi(z) \neq 0$ for $|z| > 1$. Then, using (4) we see that

$$\mathcal{M}_0(f;1) = |a_{n-\ell}|.$$

Also $\varphi(Rz) \neq 0$ for any $R > 1$. Hence, again by (4), we have

$$\mathcal{M}_0(f;R) = R^\ell \mathcal{M}_0(\varphi;R) = |a_{n-\ell}|R^{n-\ell} = \mathcal{M}_0(f;1)R^{n-\ell}.$$

Since

$$\ell \leqslant \begin{cases} n/2 & \text{if } n \text{ is even},\\ (n-1)/2 & \text{if } n \text{ is odd},\end{cases}$$

the desired result holds in this case.

Now, we shall consider the case where φ has some zeros outside the closed unit disk. Let these zeros be $\{z_j\}_{j=1}^k$, where each zero is repeated as often as its multiplicity. By (4), we have

$$\mathcal{M}_0(f;1) = |a_{n-\ell}| \prod_{j=1}^k |z_j|. \tag{14}$$

Since $f(Rz) = \sum_{\nu=\ell}^{n-\ell} a_\nu R^\nu z^\nu$, and $\{z_j/R\}_{j=1}^k$ are its only zeros whose moduli can be larger than 1, we may once again apply (4) to conclude that

$$\mathcal{M}_0(f;R) = |a_{n-\ell}| R^{n-\ell-k} \prod_{j=1}^k \left(R \cdot \max\left\{ \left| \frac{z_j}{R} \right|, 1 \right\} \right).$$

Now, we observe that $R \cdot \max\{|z_j/R|, 1\} = R|z_j/R| = |z_j|$ if $|z_j| \geqslant R$, and $R \cdot \max\{|z_j/R|, 1\} = R > |z_j|$ if $|z_j| < R$. Hence,

$$\mathcal{M}_0(f;R) \geqslant R^{n-\ell-k} \cdot |a_{n-\ell}| \prod_{j=1}^k |z_j|. \tag{15}$$

From (14) and (15) we conclude that

$$\mathcal{M}_0(f;R) \geqslant R^{n-\ell-k} \cdot \mathcal{M}_0(f;1).$$

Clearly,

$$\ell + k \leqslant \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, (11) holds also in the case where φ has some zeros outside the closed unit disk.

Inequality (11) becomes an equality for constant multiples of $z^{n/2}$ when n is even and for constant multiples of $z^{(n-1)/2}(z+1)$ in the case where n is odd. □

Remark. It seems worth mentioning that there are large classes of polynomials for which equality holds in (11).

First let n be even and ℓ a non-negative integer not exceeding $(n/2)-1$. For any given $R > 1$, let

$$P(z) := \prod_{j=1}^{(n/2)-\ell} (z - z_j) \quad \big(|z_j| \geqslant R\big). \tag{16}$$

Then, for any $a \neq 0$, the polynomial $f(z) := a z^\ell P(z) z^{(n/2)-\ell} P(1/z)$ belongs to $\wp_n$. Its degree is $n-\ell$, its highest coefficient is $a \prod_{j=1}^{(n/2)-\ell}(-z_j)$ and its zeros that lie outside the unit disk are $z_1, \ldots, z_{(n/2)-\ell}$. Hence

$$\mathcal{M}_0(f;1) = |a| \prod_{j=1}^{(n/2)-\ell} |z_j|^2.$$

Now note that the highest coefficient of $f(Rz)$ is $aR^{n-\ell} \prod_{j=1}^{(n/2)-\ell}(-z_j)$ and its zeros that lie on or outside the unit circle are $(z_1)/R, \ldots, (z_{(n/2)-\ell})/R$. Consequently

$$\mathcal{M}_0(f;R) = |a| R^{n-\ell} \prod_{j=1}^{(n/2)-\ell} \frac{|z_j|^2}{R},$$

and so

$$\mathcal{M}_0(f;R) = R^{n/2} \mathcal{M}_0(f;1).$$

Thus we see that, in the case where n is even, equality holds in (11) not only for constant multiples of $z^{n/2}$ but also for all polynomials of the form $f(z) := az^\ell P(z) z^{(n/2)-\ell} P(1/z)$, where P is as in (16).

Next, let n be odd and ℓ a non-negative integer not exceeding $(n-3)/2$. For any given $R > 1$, let

$$Q(z) := \prod_{j=1}^{\frac{n-1}{2}-\ell} (z - \zeta_j) \quad \bigl(|\zeta_j| \geqslant R\bigr). \tag{17}$$

Then, for any $b \neq 0$, the polynomial $f(z) := b(z+1) z^\ell Q(z) z^{\frac{n-1}{2}-\ell} Q(1/z)$ belongs to $\wp_n$. Its degree is $n - \ell$, its highest coefficient is $b \prod_{j=1}^{\frac{n-1}{2}-\ell} (-\zeta_j)$ and its zeros that lie on or outside the unit circle are $\zeta_1, \ldots, \zeta_{\frac{n-1}{2}-\ell}$. Hence

$$\mathcal{M}_0(f; 1) = |b| \prod_{j=1}^{\frac{n-1}{2}-\ell} |\zeta_j|^2.$$

Now note that the highest coefficient of $f(Rz)$ is $bR^{n-\ell} \prod_{j=1}^{\frac{n-1}{2}-\ell} (-\zeta_j)$ and its zeros that lie on or outside the unit circle are $(\zeta_1)/R, \ldots, (\zeta_{\frac{n-1}{2}-\ell})/R$. Consequently

$$\mathcal{M}_0(f; R) = |b| R^{n-\ell} \prod_{j=1}^{\frac{n-1}{2}-\ell} \frac{|\zeta_j|^2}{R},$$

and so

$$\mathcal{M}_0(f; R) = R^{(n+1)/2} \mathcal{M}_0(f; 1).$$

Thus we see that, in the case where n is odd, equality holds in (11) not only for constant multiples of $z^{(n-1)/2}(z+1)$ but also for all polynomials of the form $f(z) := b(z+1) z^\ell Q(z) z^{\frac{n-1}{2}-\ell} Q(1/z)$, where Q is as in (17).

Proof of Theorem 2. By a classical result of G.H. Hardy (see [9] or [11, pp. 68–69]) $\log \mathcal{M}_p(f; r)$ is a convex function of $\log r$ for any entire function f. This means that for $0 < r_1 < r_2 < r_3 < \infty$, we have

$$\frac{\log \mathcal{M}_p(f; r_3) - \log \mathcal{M}_p(f; r_2)}{\log r_3 - \log r_2} \geqslant \frac{\log \mathcal{M}_p(f; r_2) - \log \mathcal{M}_p(f; r_1)}{\log r_2 - \log r_1}.$$

In particular, if $R > 1$, then we may take $r_1 = 1/R$, $r_2 = 1$ and $r_3 = R$ in this inequality to conclude that

$$\frac{\mathcal{M}_p(f; R)}{\mathcal{M}_p(f; 1)} \geqslant \frac{\mathcal{M}_p(f; 1)}{\mathcal{M}_p(f; 1/R)}.$$

Thus, for any entire function f, we have

$$\mathcal{M}_p(f; 1) \leqslant \sqrt{\mathcal{M}_p(f; R) \mathcal{M}_p(f; 1/R)} \quad (0 < p < \infty;\ R > 1). \tag{18}$$

In the case where f belongs to $\wp_n$, that is if $f(z) \equiv z^n f(1/z)$, then

$$\mathcal{M}_p\left(f;\frac{1}{R}\right)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(\frac{1}{R}\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p\mathrm{d}\theta\right)^{1/p}=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{1}{R}\mathrm{e}^{\mathrm{i}\theta}\right)^n f\left(R\mathrm{e}^{-\mathrm{i}\theta}\right)\right|^p\mathrm{d}\theta\right)^{1/p}$$
$$=\frac{1}{R^n}\mathcal{M}_p(f;R),$$

and from (18) it follows that

$$\mathcal{M}_p(f;1)\leqslant\frac{1}{R^{n/2}}\mathcal{M}_p(f;R)\quad(R>1).$$

This proves (12).

If n is even then $f(z):=z^{n/2}$ is a polynomial belonging to $\wp_n$ for which (12) becomes an equality. $\square$

Proof of Theorem 3. Note that $f(z):=\sum_{\nu=0}^{n}a_\nu z^\nu$ belongs to $\wp_n$ if and only if $a_\nu=a_{n-\nu}$ for $\nu=0,\dots,n$. Hence,

$$\mathcal{M}_2^2(f;R)=\sum_{\nu=0}^{n}|a_\nu|^2R^{2\nu}=\frac{1}{2}\left(\sum_{\nu=0}^{n}|a_\nu|^2R^{2\nu}+\sum_{\nu=0}^{n}|a_{n-\nu}|^2R^{2\nu}\right)$$
$$=\frac{1}{2}\left(\sum_{\nu=0}^{n}|a_\nu|^2R^{2\nu}+\sum_{\nu=0}^{n}|a_\nu|^2R^{2n-2\nu}\right)=\frac{1}{2}\sum_{\nu=0}^{n}|a_\nu|^2\left(R^{2\nu}+R^{2n-2\nu}\right).$$

We claim that

$$R^{2\nu}+R^{2n-2\nu}\geqslant R^{n+1}+R^{n-1}\quad(\nu=0,\dots,n;\ R>1).\tag{19}$$

In fact, (19) can also be written in the form

$$\left(R^{n+1}-R^{2\nu}\right)\left(R^{n-1-2\nu}-1\right)\geqslant 0,$$

where the factors on the left are both non-negative for $\nu=0,\dots,\frac{n-1}{2}$ and are both non-positive for $\nu=\frac{n+1}{2},\dots,n$. Hence,

$$\mathcal{M}_2^2(f;R)=\frac{1}{2}\sum_{\nu=0}^{n}|a_\nu|^2\left(R^{2\nu}+R^{2n-2\nu}\right)\geqslant\frac{R^{n+1}+R^{n-1}}{2}\sum_{\nu=0}^{n}|a_\nu|^2$$
$$=\frac{R^{n+1}+R^{n-1}}{2}\mathcal{M}_2^2(f;1),$$

which proves (13).

In (13) equality holds for constant multiples of $z^{(n+1)/2}+z^{(n-1)/2}$. $\square$

Remark. It is easily checked that for any $f\in\wp_n$, we have

$$\mathcal{M}_p(f;r)=r^n\mathcal{M}_p\left(f;\frac{1}{r}\right)\quad(0<r<\infty;\ 0\leqslant p<\infty).$$

In view of this fact, the lower bounds for $\frac{\mathcal{M}_p(f;R)}{\mathcal{M}_p(f;1)}$, $R>1$ given in Theorems 1–3 imply corresponding lower bounds for $\frac{\mathcal{M}_p(f;\rho)}{\mathcal{M}_p(f;1)}$, $0<\rho<1$. Thus, Theorem 1 implies that *for any* $f\in\wp_n$ *and any* $\rho\in(0,1)$, *we have*

$$\mathcal{M}_0(f;\rho)\geqslant\mathcal{M}_0(f;1)\begin{cases}\rho^{n/2} & \text{if } n \text{ is even,}\\ \rho^{(n-1)/2} & \text{if } n \text{ is odd,}\end{cases}\tag{11′}$$

and that the estimate is sharp. Analogously, Theorem 2 implies that *for any* $f \in \wp_n$ *and any* $p \in (0, \infty)$*, we have*

$$\mathcal{M}_p(f;\rho) \geqslant \rho^{n/2}\mathcal{M}_p(f;1) \quad (0<\rho<1), \tag{12$'$}$$

and that the estimate is sharp for any $p \in (0,\infty)$ *and any* $\rho \in (0,1)$ *in the case where n is even.* Similarly, from Theorem 3 it follows that *if n is odd, then for any* $f \in \wp_n$*, we have*

$$\mathcal{M}_2(f;\rho) \geqslant \left(\frac{\rho^{n-1}+\rho^{n+1}}{2}\right)^{1/2}\mathcal{M}_2(f;1) \quad (0<\rho<1), \tag{13$'$}$$

and that the estimate is sharp for any $\rho \in (0,1)$.

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